

Master Equations for Gravitational Perturbations of Static Lovelock Black Holes in Higher Dimensions

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In this paper, we derive the master equations for gravitational perturbations of vector and scalar type for static vacuum Lovelock black holes. Together with our previous work on the tensor type perturbation, we now provide the full set of master equations that governs all types of gravitational perturbations of static vacuum Lovelock black holes in any dimensions.

§1. Introduction

There is a long history in black hole perturbation theory since the seminal paper by Regge and Wheeler.^{1)–3)} It is well known that there exist master equations for gravitational perturbations of static black holes in 4-dimensions.^{1)–5)} Surprisingly, the master equations have been obtained even for stationary black holes.^{3),6)} It should be mentioned that the master equations for gravitational perturbations have played an important role in gravitational wave physics.

Recently, higher dimensional black holes have attracted much attention. This is because higher dimensional black holes could be created at the LHC.⁷⁾ In fact, in the context of the braneworld with large extra-dimensions, the predicted production rate of black holes is within reach of accelerators. Another reason is that higher dimensional black holes have been used to analyze strongly coupled finite temperature field theories through the AdS/CFT correspondence. Needless to say, the master equations of gravitational perturbations of black holes in higher dimensions are crucial for the developments of these subjects.

The master equations for gravitational perturbations of higher dimensional static black holes have been obtained by Kodama and Ishibashi.⁸⁾ For stationary black holes, unfortunately, there exist only partial results. To investigate the stability of rotating black holes, a group theoretical method is developed.⁹⁾ The method is used to obtain the master equations for gravitational perturbations of squashed black holes^{10)–12)} and 5-dimensional rotating black holes with equal angular momenta.¹³⁾ The master equations for a special class of rotating black holes in more than 5-dimensions are also studied.^{14)–16)} Still, it is an open issue if the master equations exist for general rotating black holes and black rings.

We should notice that, in higher dimensions, Einstein theory is not a unique theory with the second-order differential equations. Indeed, the most general theory of gravity is Lovelock theory which is degenerated into Einstein theory in 4-dimensions. In fact, Lovelock theory is a natural extension of Einstein theory in that the Lovelock theory contains terms only up to the second order derivatives in the equations of motion. In Lovelock theory, it is known that there exist static spherical symmetric black hole solutions¹⁷⁾ and topological black hole solutions.¹⁸⁾ Hence, it is legiti-

mate to suppose black holes produced at the LHC are of this type.¹⁹⁾ Thus, it is important to obtain master equations in order to study these Lovelock black holes.

The following argument also stresses the importance of Lovelock theory. The point is that black holes are produced at the fundamental scale of higher dimensional theories. At such high energy, Einstein theory would be no longer valid. In fact, as is well known, string theory predicts Einstein theory only in the low energy limit.²⁰⁾ In string theory, there are higher curvature corrections in addition to Einstein-Hilbert term.²⁰⁾ Thus, it is natural to extend gravitational theory into those with higher power of curvature in higher dimensions. It is Lovelock theory that belongs to such class of theories.²¹⁾

In the case of second order Lovelock theory, the so-called Einstein-Gauss-Bonnet theory, the master equation for tensor perturbations has been obtained.²²⁾ The result has been also extended to the scalar and vector perturbations.²³⁾ Although Einstein-Gauss-Bonnet theory is the most general theory in five and six dimensions, it is not so in more than six dimensions. For example, when we consider ten dimensional black holes, we need to incorporate the fourth order Lovelock term. Indeed, when we consider black holes at the LHC, it is important to consider these higher order Lovelock terms.²⁴⁾ Hence, in this paper, we derive the master equations for gravitational perturbations of black holes in any order Lovelock theory, namely, in any dimensions. We have already derived the master equation for tensor perturbations of static Lovelock black holes in any dimensions.²⁵⁾ In this paper, we derive master equations for vector and scalar perturbations of static Lovelock black holes.

The organization of this paper is as follows. In Section 2, we briefly review Lovelock theory and static Lovelock black hole solutions. In section 3, we express the linear Lovelock tensor in terms of the perturbed Riemann tensor. In section 4, for completeness, we derive previous results for tensor perturbations.²⁵⁾ In section 5, we derive the master equation for vector perturbations. In Section 6, we deduce the master equation for scalar perturbations. The final section 7 is devoted to the conclusion.

§2. Static Lovelock Black Holes

In this section, we review Lovelock theory and introduce static black hole solutions.

In Ref. 21), the most general symmetric, divergence free rank (1,1) tensor is constructed out of a metric and its first and second derivatives. The corresponding Lagrangian can be constructed from m -th order Lovelock terms

$$\mathcal{L}_m = \frac{1}{2^m} \delta^{\lambda_1 \sigma_1 \dots \lambda_m \sigma_m}_{\rho_1 \kappa_1 \dots \rho_m \kappa_m} R_{\lambda_1 \sigma_1}{}^{\rho_1 \kappa_1} \dots R_{\lambda_m \sigma_m}{}^{\rho_m \kappa_m}, \quad (2.1)$$

where $R_{\lambda\sigma}{}^{\rho\kappa}$ is the Riemann tensor in D -dimensions and $\delta^{\lambda_1 \sigma_1 \dots \lambda_m \sigma_m}_{\rho_1 \kappa_1 \dots \rho_m \kappa_m}$ is the general-

ized totally antisymmetric Kronecker delta defined by

$$\delta_{\nu_1 \nu_2 \dots \nu_p}^{\mu_1 \mu_2 \dots \mu_p} = \det \begin{pmatrix} \delta_{\nu_1}^{\mu_1} & \delta_{\nu_2}^{\mu_1} & \dots & \delta_{\nu_p}^{\mu_1} \\ \delta_{\nu_1}^{\mu_2} & \delta_{\nu_2}^{\mu_2} & \dots & \delta_{\nu_p}^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\nu_1}^{\mu_p} & \delta_{\nu_2}^{\mu_p} & \dots & \delta_{\nu_p}^{\mu_p} \end{pmatrix} .$$

Then, Lovelock Lagrangian in D -dimensions is defined by

$$L = \sum_{m=0}^k c_m \mathcal{L}_m , \quad (2.2)$$

where we defined the maximum order $k \equiv [(D-1)/2]$ and c_m are arbitrary constants. Here, $[z]$ represents the maximum integer satisfying $[z] \leq z$. Hereafter, we set $c_0 = -2\Lambda$, $c_1 = a_1 = 1$ and $c_m = a_m/m$ ($m \geq 2$), for convenience. Taking variation of the Lagrangian with respect to the metric, we can derive the Lovelock equation

$$0 = \mathcal{G}_\mu^\nu = \Lambda \delta_\mu^\nu - \sum_{m=1}^k \frac{1}{2^{(m+1)}} \frac{a_m}{m} \delta_{\mu \rho_1 \kappa_1 \dots \rho_m \kappa_m}^{\nu \lambda_1 \sigma_1 \dots \lambda_m \sigma_m} R_{\lambda_1 \sigma_1}^{\rho_1 \kappa_1} \dots R_{\lambda_m \sigma_m}^{\rho_m \kappa_m} . \quad (2.3)$$

As shown in Refs. 17) and 18), there exist static exact black hole solutions of the Lovelock equations. Let us consider the following metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \bar{\gamma}_{ij} dx^i dx^j , \quad (2.4)$$

where $\bar{\gamma}_{ij}$ is the metric of $n \equiv D - 2$ -dimensional constant curvature space with a curvature $\kappa=1,0$ or -1 . Using this metric ansatz, we can calculate Riemann tensor components as

$$\begin{aligned} R_{tr}{}^{tr} &= -\frac{f''}{2} , \quad R_{ti}{}^{tj} = R_{ri}{}^{rj} = -\frac{f'}{2r} \delta_i^j , \\ R_{ij}{}^{kl} &= \left(\frac{\kappa - f}{r^2} \right) (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) . \end{aligned} \quad (2.5)$$

Substituting (2.5) into (2.3) and defining a new variable $\psi(r)$ by

$$f(r) = \kappa - r^2 \psi(r) , \quad (2.6)$$

we obtain an algebraic equation

$$W[\psi] \equiv \sum_{m=2}^k \left[\frac{a_m}{m} \left\{ \prod_{p=1}^{2m-2} (n-p) \right\} \psi^m \right] + \psi - \frac{2\Lambda}{n(n+1)} = \frac{\mu}{r^{n+1}} . \quad (2.7)$$

In (2.7), we used $n = D - 2$ and μ is a constant of integration which is related to the ADM mass of black holes as²⁶⁾

$$M = \frac{2\mu\pi^{(n+1)/2}}{\Gamma((n+1)/2)} , \quad (2.8)$$

where we used a unit $16\pi G = 1$.

In the following sections, we study gravitational perturbations around the general vacuum solutions (2.6) obtained by solving algebraic equation (2.7).

§3. Linear Lovelock Tensor

In this section, we present general formulas for the linear Lovelock tensor around the solution (2.4). The method introduced in this section is based on symmetry of the static Lovelock black holes. In detail, the point is that only $R_{tr}{}^{tr}$, $R_{ti}{}^{tj}$, $R_{ri}{}^{rj}$ and $R_{ij}{}^{kl}$ have nonzero value for the Riemann tensor due to the metric ansatz (2.4).

From now on, we use μ, ν, \dots for (t, r, x^i) , x, y, \dots for (r, x^i) , and i, j, \dots for (x^i) , respectively. With this notation, we can show that

$$\delta_{i_1 i_2 \dots i_m}^{j_1 j_2 \dots j_m} \delta_{j_1}^{i_1} = \{n - (m - 1)\} \delta_{i_2 i_3 \dots i_m}^{j_2 j_3 \dots j_m} \quad (3.1)$$

by induction. This formula is useful for later calculations. It is also easy to see that the linear Lovelock tensor reads

$$\begin{aligned} \delta \mathcal{G}_\mu^\nu = & - \sum_{m=1}^k \frac{a_m}{2^{(m+1)}} \delta_{\mu \rho_1 \kappa_1 \dots \rho_m \kappa_m}^{\nu \lambda_1 \sigma_1 \dots \lambda_m \sigma_m} \\ & \times R_{\lambda_1 \sigma_1}{}^{\rho_1 \kappa_1} \dots R_{\lambda_{m-1} \sigma_{m-1}}{}^{\rho_{m-1} \kappa_{m-1}} \delta R_{\lambda_m \sigma_m}{}^{\rho_m \kappa_m} . \end{aligned} \quad (3.2)$$

Let us explain how to calculate $\delta \mathcal{G}_t^t$ in detail and merely present final results for other components. First of all, we consider the totally antisymmetric Kronecker delta in $\delta \mathcal{G}_t^t$, namely, $\delta_{t \rho_1 \kappa_1 \dots \rho_m \kappa_m}^{t \lambda_1 \sigma_1 \dots \lambda_m \sigma_m}$. Because of the antisymmetry, t cannot show up twice in the indexes. Hence, this Kronecker delta can be rewritten as

$$\delta_{t x_1 x_2 \dots x_{2m-1} x_{2m}}^{t y_1 y_2 \dots y_{2m-1} y_{2m}} = \delta_{x_1 x_2 \dots x_{2m-1} x_{2m}}^{y_1 y_2 \dots y_{2m-1} y_{2m}} , \quad (3.3)$$

where we used properties $\delta_t^t = 1$ and $\delta_{x_p}^t = 0$. Thus, we can rewrite $\delta \mathcal{G}_t^t$ as

$$\begin{aligned} \delta \mathcal{G}_t^t = & \sum_m \left(-\frac{a_m}{2^{m+1}} \right) \delta_{x_1 x_2 \dots x_{2m-1} x_{2m}}^{y_1 y_2 \dots y_{2m-1} y_{2m}} \\ & \times R_{y_1 y_2}{}^{x_1 x_2} \dots R_{y_{2m-3} y_{2m-2}}{}^{x_{2m-3} x_{2m-2}} \delta R_{y_{2m-1} y_{2m}}{}^{x_{2m-1} x_{2m}} . \end{aligned}$$

Here, when taking the summation of x_p and y_p , we have to consider three cases. One is that there is no r index, that is, all x_p and y_p are i_p and j_p , respectively. For the other two cases, we have r index. Assume $x_p = r$, then, there must exist p' such that $y_{p'} = r$ because the totally antisymmetric delta consists of Kronecker delta. One possibility is $x_{2q-1} = r$ or $x_{2q} = r$ ($1 \leq q \leq m-1$). For this case, p' must be $p' = 2q-1$ or $2q$ because of the background property (2.5). The remaining possibility is $x_{2m-1} = r$ or $x_{2m} = r$. For this case, we have to take $p' = 2m-1$ or $2m$. The reason is as follows. If $p' \neq 2m-1$ nor $p' \neq 2m$, there must exist q' ($1 \leq q' \leq 2m-2$) such that $y_{q'} = r$. For example, if we take $y_1 = r$, x_1 or x_2 must be r because of the formula (2.5). In any case, $\delta_{x_1 x_2 \dots}^{y_1 y_2 \dots}$ must be zero because of the antisymmetry. To summarize, $\delta \mathcal{G}_t^t$ can be written as

$$\delta \mathcal{G}_t^t = \sum_m \left(-\frac{a_m}{2^{m+1}} \right) \left[\delta_{i_1 \dots i_{2m}}^{j_1 \dots j_{2m}} R_{j_1 j_2}{}^{i_1 i_2} \times \dots \times R_{j_{2m-3} j_{2m-2}}{}^{i_{2m-3} i_{2m-2}} \delta R_{j_{2m-1} j_{2m}}{}^{i_{2m-1} i_{2m}} \right]$$

$$\begin{aligned}
& + 4(m-1)\delta_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} \\
& \quad \times R_{rj_1}^{ri_1} R_{j_2 j_3}^{i_2 i_3} \times \dots \times R_{j_{2m-4} j_{2m-3}}^{i_{2m-4} i_{2m-3}} \delta R_{j_{2m-2} j_{2m-1}}^{i_{2m-2} i_{2m-1}} \\
& + 4\delta_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} R_{j_1 j_2}^{i_1 i_2} \times \dots \times R_{j_{2m-3} j_{2m-2}}^{i_{2m-3} i_{2m-2}} \delta R_{j_{2m-1} r}^{i_{2m-1} r} \Bigg] . \quad (3.4)
\end{aligned}$$

Substituting the background quantities (2.5) into (3.4) and using the formula (3.1), we can proceed as

$$\begin{aligned}
\delta \mathcal{G}_t^t &= \sum_{m=1}^k \left(-\frac{a_m}{2^{m+1}} \right) \left[\right. \\
& \quad 2^{m-1} \left(\frac{\kappa - f}{r^2} \right)^{m-1} \delta_{i_1 \dots i_{2m}}^{j_1 \dots j_{2m}} \delta_{j_1}^{i_1} \dots \delta_{j_{2m-2}}^{i_{2m-2}} \delta R_{j_{2m-1} j_{2m}}^{i_{2m-1} i_{2m}} \\
& \quad + 4(m-1)2^{m-2} \left(-\frac{f'}{2r} \right) \left(\frac{\kappa - f}{r^2} \right)^{m-2} \\
& \quad \quad \times \delta_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} \delta_{j_1}^{i_1} \dots \delta_{j_{2m-3}}^{i_{2m-3}} \delta R_{j_{2m-2} j_{2m-1}}^{i_{2m-2} i_{2m-1}} \\
& \quad \left. + 4 \cdot 2^{m-1} \left(\frac{\kappa - f}{r^2} \right)^{m-1} \delta_{i_1 \dots i_{2m-1}}^{j_1 \dots j_{2m-1}} \delta_{j_1}^{i_1} \dots \delta_{j_{2m-2}}^{i_{2m-2}} \delta R_{j_{2m-1} r}^{i_{2m-1} r} \right] \\
&= \sum_{m=1}^k a_m \left[\left(-\frac{1}{4} \left\{ \prod_{p=2}^{2m-1} (n-p) \right\} \right) \left(\frac{\kappa - f}{r^2} \right)^{m-1} \right. \\
& \quad + (m-1) \left\{ \prod_{p=2}^{2m-2} (n-p) \right\} \left(\frac{f'}{4r} \right) \left(\frac{\kappa - f}{r^2} \right)^{m-2} \delta_{ik}^{jl} \delta R_{jl}^{ik} \\
& \quad \left. - \left\{ \prod_{p=1}^{2m-2} (n-p) \right\} \left(\frac{\kappa - f}{r^2} \right)^{m-1} \delta_i^j \delta R_{jr}^{ir} \right] \\
&= -\frac{T'}{2(n-1)r^{n-2}} \delta_i^j \delta_k^l \delta R_{jl}^{ik} - \frac{T}{r^{n-1}} \delta_i^j \delta R_{jr}^{ir} , \quad (3.5)
\end{aligned}$$

where we used the relation $\delta_{ij}^{kl} \delta R_{kl}^{ij} = 2\delta_i^k \delta_j^l \delta R_{kl}^{ij}$ in the last equality. Here, $T(r)$ is defined by

$$T(r) \equiv r^{n-1} \partial_\psi W[\psi] = r^{n-1} \left(1 + \sum_{m=2}^k \left[a_m \left\{ \prod_{p=1}^{2m-2} (n-p) \right\} \psi^{m-1} \right] \right) . \quad (3.6)$$

Similarly, we can deduce other components of the linear Lovelock tensor. The results are as follows;

$$\delta \mathcal{G}_t^t = -\frac{T'}{2(n-1)r^{n-2}} \delta_i^j \delta_k^l \delta R_{jl}^{ik} - \frac{T}{r^{n-1}} \delta_i^j \delta R_{jr}^{ir} ,$$

$$\begin{aligned}
\delta\mathcal{G}_t^r &= -\frac{T}{r^{n-1}}\delta_i^j\delta R_{jt}^{ir} , \\
\delta\mathcal{G}_t^i &= \frac{T'}{(n-1)r^{n-2}}\delta_k^j\delta R_{tj}^{ik} + \frac{T}{r^{n-1}}\delta R_{tr}^{ir} , \\
\delta\mathcal{G}_r^r &= -\frac{T'}{2(n-1)r^{n-2}}\delta_i^j\delta_k^l\delta R_{jl}^{ik} - \frac{T}{r^{n-1}}\delta_i^j\delta R_{jt}^{it} , \\
\delta\mathcal{G}_r^i &= \frac{T'}{(n-1)r^{n-2}}\delta_k^j\delta R_{rj}^{ik} + \frac{T}{r^{n-1}}\delta R_{rt}^{it} , \\
\delta\mathcal{G}_i^j &= \frac{T'}{(n-1)r^{n-2}}(\delta R_{ti}^{tj} + \delta R_{ri}^{rj}) + \frac{T''}{(n-1)(n-2)r^{n-3}}\delta_l^k\delta R_{ik}^{jl} \\
&\quad - \delta_i^j \left[\frac{T}{r^{n-1}}\delta R_{tr}^{tr} + \frac{T'}{(n-1)r^{n-2}}(\delta R_{tk}^{tl} + \delta R_{rk}^{rl})\delta_l^k \right. \\
&\quad \left. + \frac{T''}{2(n-1)(n-2)r^{n-3}}\delta_l^k\delta_p^q\delta R_{kq}^{lp} \right] . \tag{3.7}
\end{aligned}$$

Thus, in order to derive the linear Lovelock equations, we need to know the perturbed Riemann tensor $\delta R_{\mu\nu}{}^{\rho\lambda}$.

Since Lovelock black holes have n -dimensional symmetric space, we can classify perturbations into tensor, vector and scalar type perturbations. In following sections, we treat these three type of perturbations separately.

§4. Master equation for tensor perturbations

In this section, we derive the master equation for tensor perturbations of static Lovelock black holes.

Tensor perturbations around the solution (2.4) is characterized by

$$\delta g_{ab} = 0 , \quad \delta g_{ai} = 0 , \quad \delta g_{ij} = r^2\phi(t, r)\bar{h}_{ij}(x^i) , \tag{4.1}$$

where $a, b = (t, r)$ and $\phi(t, r)$ represents the dynamical degrees of freedom. Here, tensor harmonics \bar{h}_{ij} are defined by

$$\bar{\nabla}^k\bar{\nabla}_k\bar{h}_{ij} = -\gamma_t\bar{h}_{ij} , \quad \bar{\nabla}^i\bar{h}_{ij} = 0 , \quad \bar{\gamma}^{ij}\bar{h}_{ij} = 0 , \tag{4.2}$$

where $\bar{\nabla}^i$ denotes a covariant derivative with respect to $\bar{\gamma}_{ij}$ and the eigenvalue is given by $\gamma_t = \ell(\ell + n - 1) - 2$, ($\ell = 2, 3, 4, \dots$) for $\kappa = 1$ and positive real numbers for $\kappa = -1, 0$. Note that indexes i, j, \dots are raised or lowered by $\bar{\gamma}_{ij}$.

For tensor perturbations, from (3.7), it is clear that components other than $\delta\mathcal{G}_i^j$ vanish and the terms proportional to δ_i^j in $\delta\mathcal{G}_i^j$ also disappear. Therefore, what we have to calculate are δR_{ti}^{tj} , δR_{ri}^{rj} and $\delta_l^m\delta R_{im}^{jl}$. From the metric ansatz (2.4) and (4.1), these components can be deduced as

$$\delta R_{ti}^{tj} = \left[\frac{\ddot{\phi}}{2f} - \frac{f'\phi'}{4} \right] \bar{h}_i^j ,$$

$$\begin{aligned}\delta R_{ri}{}^{rj} &= \left[-\frac{f\phi''}{2} + \left(-\frac{f'}{4} - \frac{f}{r} \right) \phi' \right] \bar{h}_i{}^j, \\ \delta_l^m \delta R_{im}{}^{jl} &= \left[-\frac{n-2}{2} \frac{f}{r} \phi' + \frac{2\kappa + \gamma_t}{2r^2} \phi \right] \bar{h}_i{}^j.\end{aligned}\quad (4.3)$$

Then, substituting these results into (3.7), we can calculate $\delta\mathcal{G}_i^j$ as follows;

$$\begin{aligned}(n-1)r^{n-2}\delta\mathcal{G}_i^j &= T' (\delta R_{ti}{}^{tj} + \delta R_{ri}{}^{rj}) + \frac{rT''}{(n-2)} \delta_l^m \delta R_{im}{}^{jl} \\ &= \left[\frac{T'}{2f} (\ddot{\phi} - f^2\phi'') - \left(\frac{f'T'}{2} + \frac{fT'}{r} + \frac{fT''}{2} \right) \phi' + \frac{(2\kappa + \gamma_t)T''}{2(n-2)r} \phi \right] \bar{h}_i{}^j.\end{aligned}\quad (4.4)$$

Separating the variables $\phi(r, t) = \chi(r)e^{-i\omega t}$, we can derive the master equation for the tensor perturbations from the linear Lovelock equation $\delta\mathcal{G}_\mu^\nu = 0$ as follows;

$$-f^2\chi'' - \left(f^2\frac{T''}{T'} + \frac{2f^2}{r} + ff' \right) \chi' + \frac{(2\kappa + \gamma_t)f}{(n-2)r} \frac{T''}{T'} \chi = \omega^2\chi. \quad (4.5)$$

Here, we should stress that we have assumed nothing for Lovelock coefficients and $f(r)$. Hence, the master equation we derived is quite general.

Furthermore, introducing a new function $\Psi(r) = \chi(r)r\sqrt{T'(r)}$ and using tortoise coordinate r^* which is defined as $dr^*/dr = 1/f(r)$, we can transform the master equation (4.5) into Schrödinger type equation

$$-\frac{d^2\Psi}{dr^{*2}} + V_t(r(r^*))\Psi = \omega^2\Psi, \quad (4.6)$$

where we have defined the effective potential

$$V_t(r) = \frac{(2\kappa + \gamma_t)f}{(n-2)r} \frac{d \ln T'}{dr} + \frac{1}{r\sqrt{T'}} f \frac{d}{dr} \left(f \frac{d}{dr} r \sqrt{T'} \right). \quad (4.7)$$

Here, we have assumed $T' > 0$ in order to avoid ghost instability.²⁵⁾

§5. Master equation for vector perturbations

In this section, we consider vector perturbations of static Lovelock black holes and derive the master equation.

We take the Regge-Wheeler gauge

$$\delta g_{\mu\nu} = \left(\begin{array}{cc|c} 0 & 0 & v_i \\ 0 & 0 & w_i \\ \hline \text{sym} & \text{sym} & 0 \end{array} \right), \quad (5.1)$$

where v_i and w_i satisfy transverse condition $\bar{\nabla}^i v_i = \bar{\nabla}^i w_i = 0$ and “sym” means symmetric part of metric perturbations.

From the ansatz (5.1), it is clear that $\delta\mathcal{G}_t^t = \delta\mathcal{G}_r^r = \delta\mathcal{G}_t^r = 0$. The components of $\delta R_{\mu\nu}{}^{\rho\lambda}$ which is necessary for calculations of non-zero components of $\delta\mathcal{G}_\mu^\nu$ are as follows;

$$\begin{aligned}
\delta R_{tk}{}^{ij} &= \frac{\kappa - f}{r^4} \left(\delta_k^j v^i - \delta_k^i v^j \right) + \frac{f'}{2r^3} \left(\delta_k^j v^i - \delta_k^i v^j \right) \\
&\quad - \frac{1}{2r^4} \left[-v^j|_k{}^i + v^i|_k{}^j + \kappa(\delta_k^j v^i - \delta_k^i v^j) \right] \\
&\quad - \frac{f}{2r^3} \left[\delta_k^j \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right) - \delta_k^i \left(v^{j'} - \dot{w}^j - \frac{2}{r} v^j \right) \right], \\
\delta R_{tr}{}^{ir} &= -\frac{f'}{2r^3} v^i + \frac{f''}{2r^2} v^i \\
&\quad - \frac{f}{2r^3} \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right) - \frac{f}{2r^2} \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right)', \\
\delta R_{rk}{}^{ij} &= \frac{\kappa - f}{r^4} \left(\delta_k^j w^i - \delta_k^i w^j \right) + \frac{f'}{2r^3} \left(\delta_k^j w^i - \delta_k^i w^j \right) \\
&\quad + \frac{1}{2r^4} \left[w^j|_k{}^i - w^i|_k{}^j - \kappa(\delta_k^j w^i - \delta_k^i w^j) \right], \\
\delta R_{tr}{}^{ti} &= -\frac{f'}{2r^3} w^i + \frac{f''}{2r^2} w^i - \frac{1}{2r^2 f} \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right), \\
\delta R_{ti}{}^{tj} &= -\frac{1}{2r^2 f} \left(\dot{v}_i{}^j + \dot{v}^j|_i \right) + \frac{f'}{4r^2} \left(w_i{}^j + w^j|_i \right), \\
\delta R_{ri}{}^{rj} &= \frac{f'}{4r^2} \left(w_i{}^j + w^j|_i \right) + \frac{f}{2r^2} \left(w_i{}^j + w^j|_i \right)', \\
\delta R_{ij}{}^{kl} &= \frac{f}{2r^3} \left[\delta_j^l \left(w_i{}^k + w^k|_i \right) - \delta_j^k \left(w_i{}^l + w^l|_i \right) \right. \\
&\quad \left. + \delta_i^k \left(w_j{}^l + w^l|_j \right) - \delta_i^l \left(w_j{}^k + w^k|_j \right) \right], \tag{5.2}
\end{aligned}$$

where we use $|i$ as a covariant derivative with respect to $\bar{\gamma}_{ij}$ instead of $\bar{\nabla}_i$. Then, substituting these results into (3.7), we can get the linear Lovelock tensor as follows:

$$\begin{aligned}
2r^{n+2} \delta\mathcal{G}_t^i &= -\frac{T'}{n-1} v^i|_k{}^k - \kappa T' v^i - f \left\{ rT \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right) \right\}', \\
2r^{n+2} \delta\mathcal{G}_r^i &= -\frac{T'}{n-1} w^i|_k{}^k - \kappa T' w^i - \frac{rT}{f} \left(v^{i'} - \dot{w}^i - \frac{2}{r} v^i \right), \\
2(n-1)r^n \delta\mathcal{G}_i^j &= -\frac{T'}{f} \left(\dot{v}_i{}^j + \dot{v}^j|_i \right) + \left\{ fT' \left(w_i{}^j + w^j|_i \right) \right\}', \tag{5.3}
\end{aligned}$$

where we used the relation

$$\{rf' + 2(\kappa - f)\}T = (n+1)\mu, \tag{5.4}$$

and its derivative with respect to r . Then, after expanding metric perturbations by vector harmonics \bar{V}_i , which satisfies $\bar{\nabla}_k \bar{\nabla}^k \bar{V}_i = -\gamma_v \bar{V}_i$ with $\gamma_v = \ell(\ell + n - 1) - 1$ ($\ell \geq 2$) for $\kappa = 1$ and non-negative numbers for $\kappa = 0, -1$, we can get the linear Lovelock equation as

$$\begin{cases} \left(\frac{\gamma_v}{n-1} - \kappa \right) v T' - f \left\{ r T \left(v' - \dot{w} - \frac{2}{r} v \right) \right\}' = 0, \\ \left(\frac{\gamma_v}{n-1} - \kappa \right) w T' - \frac{r T}{f} \left(v' - \dot{w} - \frac{2}{r} v \right)' = 0, \\ -\frac{T'}{f} \dot{v} + (f T' w)' = 0. \end{cases} \quad (5.5)$$

Note that only two equations among these equations are independent. In fact, we can get the third equation in (5.5) by combining a derivative of the first equation with respect to t and a derivative of the second equation multiplied by $f(r)$ with respect to r . Here, we use the second and third equations in (5.5). We have not considered the exceptional mode $\gamma_v = (n-1)\kappa$ since its treatment is well known (8, 27).

In this paper, we assume T' is always positive otherwise there exists ghost instability for tensor perturbations. Under this assumption, we can eliminate v from the second equation in (5.5) using the third equation. The resultant master equation is given by

$$\frac{r T}{f} \ddot{w} - \frac{r^3 T}{f} \left\{ \frac{f}{r^2 T'} (f T' w)' \right\}' + \left(\frac{\gamma_v}{n-1} - \kappa \right) T' w = 0. \quad (5.6)$$

Furthermore, under the assumption of positivity of T' , we can introduce a new variable

$$\chi = \frac{f}{r} \sqrt{T'} w.$$

We also perform Fourier transformation $\chi = \Psi e^{-i\omega t}$. Thus, using the new variable and a tortoise coordinate r^* , we can rewrite Eq. (5.6) as

$$-\partial_{r^*}^2 \Psi + V_v(r) \Psi = \omega^2 \Psi \quad (5.7)$$

where we have defined the effective potential for vector perturbations

$$V_v(r) = r \sqrt{T'} f \partial_r \left(f \partial_r \frac{1}{r \sqrt{T'}} \right) + \left(\frac{\gamma_v}{n-1} - \kappa \right) \frac{f T'}{r T}. \quad (5.8)$$

Note that we have not assumed anything for $f(r)$ and Lovelock coefficients a_m except for the positivity of T' . In this sense, this Schrödinger type master equation is quite general.

§6. Master equation for Scalar Perturbations

In this section, we derive the master equation for scalar perturbations of static Lovelock black holes.

We use the Zerilli gauge:

$$h_{\mu\nu} = \left(\begin{array}{cc|c} f\bar{H} & H_1 & 0 \\ sym & H/f & 0 \\ \hline \mathbf{sym} & \mathbf{sym} & r^2 K \bar{\gamma}_{ij} \end{array} \right), \quad (6.1)$$

where “sym” represents symmetric part of metric perturbations.

Following Ref. 23), we derive the master equation from $\delta\mathcal{G}_i^j = 0 (i \neq j)$, $\delta\mathcal{G}_t^r = 0$, $\delta\mathcal{G}_t^t = 0$, $\delta\mathcal{G}_r^i = 0$ and $\delta\mathcal{G}_r^r = 0$. Taking look at Eqs.(3.7), we see that we need $\delta R_{ti}{}^{tj}$, $\delta R_{ri}{}^{rj}$, $\delta R_{ij}{}^{kl}$, $\delta R_{ti}{}^{rj}$, $\delta R_{ri}{}^{jk}$ and $\delta R_{tr}{}^{ti}$. With the Zerilli gauge, these components can be calculated as follows:

$$\begin{aligned} \delta R_{ti}{}^{tj} &= \frac{1}{2r^2} \bar{H}_{|i}{}^{|j} + \left(-\frac{1}{r} \dot{H}_1 + \frac{1}{2f} \ddot{K} - \frac{f'}{4} K' + \frac{f}{2r} \bar{H}' + \frac{f'}{2r} H \right) \delta_i^j, \\ \delta R_{ri}{}^{rj} &= -\frac{1}{2r^2} H_{|i}{}^{|j} + \left(\frac{1}{2r} (fH)' - \frac{f'}{4} K' - \frac{f}{r} K' - \frac{f}{2} K'' \right) \delta_i^j, \\ \delta R_{ij}{}^{kl} &= \frac{1}{2r^2} \left[K_{|j}{}^{|k} \delta_i^l - K_{|i}{}^{|k} \delta_j^l - K_{|j}{}^{|l} \delta_i^k + K_{|i}{}^{|l} \delta_j^k \right] \\ &\quad + \left(-\frac{\kappa}{r^2} K + \frac{f}{r^2} H - \frac{f}{r} K' \right) \left(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k \right), \\ \delta R_{ti}{}^{rj} &= -\frac{f}{2r^2} H_{1|i}{}^{|j} - \left(-\frac{f}{2r} \dot{H} - \frac{f'}{4} \dot{K} + \frac{f}{2} \dot{K}' + \frac{f}{2r} \dot{K} \right) \delta_i^j, \\ \delta R_{ri}{}^{jk} &= \frac{1}{2r^3} \left(H^{[j} \delta_i^k - H^{[k} \delta_i^j \right) - \frac{1}{2r^2} \left(K'^{[j} \delta_i^k - K'^{[k} \delta_i^j \right), \\ \delta R_{tr}{}^{ti} &= \frac{1}{2r^2} \left[\frac{f'}{2f} (H + \bar{H}) + \bar{H}' - \frac{1}{r} \bar{H} - \frac{1}{f} \dot{H}_1 \right]^{|i}, \end{aligned} \quad (6.2)$$

where $|i$ means a covariant derivative with respect to $\bar{\gamma}_{ij}$. Then, we can calculate the linear Lovelock equations. Expanding metric perturbations in terms of scalar harmonics \bar{Y} which satisfy $\bar{\nabla}_k \bar{\nabla}^k \bar{Y} = -\gamma_s \bar{Y}$ with $\gamma_s = \ell(\ell + n - 1)$ for $\kappa = 1$ and positive numbers for $\kappa = 0, -1$, we obtain the linear Lovelock equations

$$\delta\mathcal{G}_i^j = 0 \ (i \neq j) \Leftrightarrow T'(\bar{H} - H) = rT''K, \quad (6.3)$$

$$\delta\mathcal{G}_t^r = 0 \Leftrightarrow -\gamma_s H_1 + n \left[r(K - H) + r^2 K' - \frac{r^2 f'}{2f} K \right] = 0, \quad (6.4)$$

$$\begin{aligned} \delta\mathcal{G}_t^t = 0 \Leftrightarrow & \left\{ -\gamma_s T - nr(fT)' \right\} H - nTrfH' + rT'(n\kappa - \gamma_s)K \\ & + \left\{ \frac{nTr^2 f'}{2} + nf(r^2 T)' \right\} K' + nTr^2 fK'' = 0, \end{aligned} \quad (6.5)$$

$$\begin{aligned} \delta\mathcal{G}_r^i = 0 \Leftrightarrow rT' \left(\frac{1}{r}H - K' \right) \\ + T \left[\frac{f'}{2f}(H + \bar{H}) + \bar{H}' - \frac{1}{r}\bar{H} - \frac{1}{f}\dot{H}_1 \right] = 0 , \end{aligned} \quad (6.6)$$

$$\begin{aligned} \delta\mathcal{G}_r^r = 0 \Leftrightarrow \frac{nTr^2}{f} \left(\frac{2f}{r}\dot{H}_1 - \ddot{K} \right) + rT'(n\kappa - \gamma_s)K + \left(nT'r^2f + \frac{nTr^2f'}{2} \right) K' \\ - nr(Tf)'H + \gamma_s T\bar{H} - nTrf\bar{H}' = 0 . \end{aligned} \quad (6.7)$$

Now we derive the master equation for scalar perturbations from these equations. We do not consider exceptional gauge dependent modes $\gamma_s = 0$ and $\gamma_s = n\kappa$ modes since treatment of these modes is well known.^{8),27)} We also assume $T' > 0$ outside the horizon again. First of all, we show that all perturbative variables \bar{H} , H_1 , H and K can be expressed by a single master function ϕ defined later. From Eq. (6.3), \bar{H} can be expressed by H and K as

$$\bar{H} = H + \frac{rT''}{T'}K . \quad (6.8)$$

By inspecting Eq. (6.4), we see that it is convenient to define a master function ϕ as

$$H_1 \equiv \frac{r}{f} \left(\dot{\phi} + \dot{K} \right) . \quad (6.9)$$

Then, we can express H using ϕ and K by integrating (6.4) with respect to t . The result is given by

$$H = -\frac{\gamma_s}{nf}\phi + rK' - \frac{A(r)}{2nf}K , \quad (6.10)$$

where

$$A(r) = -2nf + 2\gamma_s + nrf' . \quad (6.11)$$

Note that there may exist an arbitrary function of r as a constant of integration. However, this function can be absorbed into the definition of ϕ . From Eqs.(6.8), (6.9) and (6.10), it turns out that we need to express K by ϕ in order to express all variables in terms of ϕ . Substituting (6.10) into (6.5), we obtain such a formula

$$K = -\frac{2}{A} \left[nrf\phi' + \left(\gamma_s + nrf\frac{T'}{T} \right) \phi \right] , \quad (6.12)$$

where we used a relation

$$(AT)' = 2\gamma_s T' + n\{(rf' - 2f)T\}' = 2(\gamma_s - n\kappa)T' , \quad (6.13)$$

which can be derived from (5.4).

Now, we are in a position to derive the master equation for the master variable ϕ . From Eqs.(6.6) and (6.7), we can make the following combination

$$\begin{aligned} & nfr \times (\text{l.h.s of (6.6)}) + (\text{l.h.s of (6.7)}) \\ &= \frac{AT}{2} \bar{H} - (\gamma_s - n\kappa) r T' K + \frac{nTrf'}{2} (rK' - H) + \frac{nTr^2}{f} \left(\frac{f}{r} \dot{H}_1 - \ddot{K} \right). \end{aligned} \quad (6.14)$$

Substituting (6.8), (6.9), (6.10) and (6.12) into the above equation (6.14), we get the master equation for scalar perturbations

$$\ddot{\phi} - f^2 \phi'' + f^2 \left(\ln \left(\frac{A^2}{r^2 f T'} \right) \right)' \phi' + Q \phi = 0, \quad (6.15)$$

where we have defined

$$Q = \frac{f}{nTr^2} \left[\left(2 \frac{(AT)'}{AT} - \frac{T''}{T'} \right) (\gamma_s r T + n r^2 f T') - n (r^2 f T')' \right]. \quad (6.16)$$

Finally, we change the normalization of ϕ as

$$\phi = N(r) \chi, \quad N = \frac{A}{r \sqrt{T'}}. \quad (6.17)$$

It is also convenient to move on to Fourier space as $\chi = \Psi e^{-i\omega t}$. Substituting these into the master equation (6.15) and using a tortoise coordinate r^* , we obtain Schrödinger type equation

$$-\partial_{r^*}^2 \Psi + V_s(r) \Psi = \omega^2 \Psi, \quad (6.18)$$

where we have defined the effective potential for scalar perturbations

$$\begin{aligned} V_s(r) &= 2\gamma_s f \frac{(rNT)'}{nNT r^2} \\ &\quad - \frac{f}{N} \partial_r (f \partial_r N) + 2f^2 \frac{N'^2}{N^2} - \frac{f}{T} \partial_r (f \partial_r T) + 2f^2 \frac{T'^2}{T^2} + 2f^2 \frac{N' T'}{NT}. \end{aligned} \quad (6.19)$$

We assumed that T' is always positive because tensor perturbation has ghost instability if this assumption is not fulfilled. However, except for this assumption, we have not imposed any conditions on the Lovelock coefficient a_m and $f(r)$. Therefore, the master equation we have derived is again quite general.

§7. Conclusion

We have succeeded in deriving master equations for gravitational perturbations of static Lovelock black holes. The results can be regarded as a generalization of master equations in Einstein theory derived by Kodama and Ishibashi. Of course, it is possible to extend our analysis to charged black holes and other black holes.^{28), 29)} Moreover, our result would serve a starting point for studying stationary black holes

in Lovelock theory. It is also interesting to consider Euclidean version of our results in the light of black hole thermodynamics.³⁰⁾

There are many applications of our master equations. In an accompanying paper, using the master equations, we show that static Lovelock black holes with small masses are unstable in the asymptotically flat cases.³¹⁾ The application to asymptotically AdS cases is also interesting from the point of view of the AdS/CFT correspondence,^{32)–34)} in particular, in relation to stability of holographic superconductors.^{35), 36)}

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